

# Quintom model with $O(N)$ symmetry

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## Abstract

We investigate the quintom model of dark energy in the generalized case where the corresponding canonical and phantom fields possess  $O(N)$  symmetries. Assuming exponential potentials we find that this  $O(N)$  quintom paradigm exhibits novel properties comparing to the simple canonical and phantom scenarios. In particular, we find that the universe cannot result in a quintessence-type solution with  $w > -1$ , even in the cases where the phantom field seems to be irrelevant. On the contrary, there are always late-time attractors which correspond to accelerating universes with  $w < -1$  and with a recent crossing of the phantom divide, and for a very large area of the parameter space they are the only ones. This is in contrast with the previous simple-quintom results, where an accelerating universe is a possible late-time stable solution but it is not guaranteed.

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# 1 Introduction

Many cosmological observations, such as SNe Ia [1], WMAP [2], SDSS [3], Chandra X-ray observatory [4], etc., reveal that our universe is undergoing an accelerating expansion. In addition, they suggest that it is spatially flat, and consists of about 70% of dark energy with negative pressure, of 30% of dust matter (cold dark matter plus baryons), and of an negligible amount of radiation. Although the nature and origin of dark energy could perhaps understood by a fundamental underlying theory unknown up to now, physicists can still propose some paradigms to describe it. The most obvious theoretical candidate for dark energy is the cosmological constant [5, 6, 7] which has the equation of state  $w = -1$ . However, it leads to the two known difficulties [8], namely the “fine-tuning” problem (why is the current vacuum energy density so small), and the “cosmic coincidence” one (why are the densities of vacuum energy and dark matter nearly equal today since they scale very differently during the expansion history).

There have been many efforts to resolve these problems [6, 9], but none could offer a robust and undoubted solution. This fact led many theoretical physicists to construct alternative frameworks, such is the dynamical dark energy scenario, by assuming that the vacuum energy is cancelled to exactly zero by some unknown mechanism, and introducing a dark energy component with a dynamically variable equation of state. The dynamical dark energy proposal is often realized by some scalar-field mechanism which suggests that the energy form with negative pressure is provided by a scalar field evolving downwards a proper potential. A large class of scalar-field dark energy models have been studied in the literature, including quintessence [10], K-essence [11], tachyon [12], phantom [13], ghost condensate [14], holographic dark energy [15], bulk holographic dark energy [16] and many others.

A primary scalar field candidate for dark energy was the quintessence scenario [10], which consists of a fluid with equation-of-state parameter lying in the range  $-1 < w < -1/3$ . On the other hand, for the phantom model [13] of dark energy, which consists of a scalar field with a negative sign of the kinetic term in the Lagrangian, one always obtains  $w \leq -1$ . Thus, neither the quintessence nor the phantom alone can fulfill the transition from  $w > -1$  to  $w < -1$  and vice versa, although the cosmological observations mildly favor models where such a transition was indeed realized and in particular with  $w$  crossing  $-1$  in the near past. As it was indicated in the literature [17], the consideration of the combination of quintessence and phantom in a unified model, leads to the fulfillment of the aforementioned transition through the  $w = -1$  divide. This model, dubbed quintom, can produce a better fit to the observational data.

The generalization of quintessence and phantom models to fields with  $O(N)$  symmetry have been performed in [18] and [19] respectively. As it was shown, the behavior of the corresponding dynamical systems, in specific areas of the parameter space, can be qualitatively different than then single-field models. In the present work we are interested in investigating the generalized quintom model with an  $O(N)$  symmetry. As a specific potential form we impose the exponential dependence on the corresponding fields, since exponential potentials are known to be significant in various cosmological models [20]. We perform a complete phase-space stability analysis of the corresponding autonomous system and we extract its attractor properties. We find that the  $O(N)$  quintom has qualitatively novel properties compared to the corresponding simple

phantom, quintessence and quintom models. In particular, cosmological solutions with  $w < -1$  are the only attractors for a very large area of the parameter space, and the crossing through the  $w = -1$  divide has a large probability to be realized.

The plan of the work is as follows: In section 2 we construct the quintom model with  $O(N)$  symmetries and in section 3 we perform its complete stability analysis. In section 4 we discuss the cosmological implications of our results, and finally section 5 is devoted to conclusions.

## 2 $O(N)$ quintom

We consider a flat Robertson-Walker metric:

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2. \quad (1)$$

The Lagrangian density for a quintom model with  $O(N)$  symmetries is:

$$L = \frac{1}{2}g^{\mu\nu} \left[ (\partial_\mu \Phi^\alpha)(\partial_\nu \Phi^\alpha) - (\partial_\mu \sigma^\beta)(\partial_\nu \sigma^\beta) \right] - V_\Phi(|\Phi^\alpha|) - V_\sigma(|\sigma^\beta|), \quad (2)$$

where  $\Phi^\alpha$  is the component of the canonical field, with  $\alpha = 1, 2, \dots, N_\Phi$ , and  $\sigma^\beta$  is the component of the phantom field, with  $\beta = 1, 2, \dots, N_\sigma$ . Note that in the general case, the dimensionality of the multiplets of the two fields, is not the same. Fortunately, these dimensionalities do not appear in the final form of the equations.

In order to impose the  $O(N)$  symmetries, following [18], we write:

$$\begin{aligned} \Phi^1 &= R_\Phi(t) \cos \varphi_{\Phi_1}(t) \\ \Phi^2 &= R_\Phi(t) \sin \varphi_{\Phi_1}(t) \cos \varphi_{\Phi_2}(t) \\ \Phi^3 &= R_\Phi(t) \sin \varphi_{\Phi_1}(t) \sin \varphi_{\Phi_2}(t) \cos \varphi_{\Phi_3}(t) \\ &\dots\dots\dots \\ \Phi^{N_\Phi-1} &= R_\Phi(t) \sin \varphi_{\Phi_1}(t) \dots \sin \varphi_{\Phi_{N_\Phi-2}}(t) \cos \varphi_{\Phi_{N_\Phi-1}}(t) \\ \Phi^{N_\Phi} &= R_\Phi(t) \sin \varphi_{\Phi_1}(t) \dots \sin \varphi_{\Phi_{N_\Phi-2}}(t) \sin \varphi_{\Phi_{N_\Phi-1}}(t), \end{aligned} \quad (3)$$

and similarly:

$$\begin{aligned} \sigma^1 &= R_\sigma(t) \cos \varphi_{\sigma_1}(t) \\ \sigma^2 &= R_\sigma(t) \sin \varphi_{\sigma_1}(t) \cos \varphi_{\sigma_2}(t) \\ \sigma^3 &= R_\sigma(t) \sin \varphi_{\sigma_1}(t) \sin \varphi_{\sigma_2}(t) \cos \varphi_{\sigma_3}(t) \\ &\dots\dots\dots \\ \sigma^{N_\sigma-1} &= R_\sigma(t) \sin \varphi_{\sigma_1}(t) \dots \sin \varphi_{\sigma_{N_\sigma-2}}(t) \cos \varphi_{\sigma_{N_\sigma-1}}(t) \\ \sigma^{N_\sigma} &= R_\sigma(t) \sin \varphi_{\sigma_1}(t) \dots \sin \varphi_{\sigma_{N_\sigma-2}}(t) \sin \varphi_{\sigma_{N_\sigma-1}}(t). \end{aligned} \quad (4)$$

Thus, we have explicitly used the properties  $|\Phi^\alpha| = R_\Phi$  and  $|\sigma^\beta| = R_\sigma$ . Furthermore, we assume that the potentials  $V_\Phi(|\Phi^\alpha|)$  and  $V_\sigma(|\sigma^\beta|)$  depend only on  $R_\Phi$  and  $R_\sigma$  respectively.

The action for the universe is as usual:

$$S = \int d^4x \sqrt{-g} \left( -\frac{1}{16\pi G} R_s - p_\gamma + L \right), \quad (5)$$

where  $g$  is the determinant of the metric tensor  $g_{\mu\nu}$ ,  $R_s$  is the Ricci scalar and  $G$  is the Newton's constant (in the following we will instead use  $\kappa^2 \equiv 8\pi G$ ).  $p_\gamma$  is the pressure of the barotropic fluid which constitutes the matter content of the universe, with equation of state  $p_\gamma = (\gamma - 1)\rho_\gamma$ , with the constant  $\gamma$  in the interval  $0 < \gamma < 2$ .

The Einstein equations for the angles  $\varphi_{\Phi_\alpha}$  and  $\varphi_{\sigma_\beta}$  can be easily derived through the corresponding variation of the action. However, they are irrelevant for the purpose of this work since we are going to use only the radial equations for the fields, i.e. the equations determining the evolution of  $R_\Phi(t)$  and  $R_\sigma(t)$ , plus the Friedmann equations. Hence, we have:

$$H^2 = \frac{\kappa^2}{3} [\rho_\gamma + \rho_\Phi + \rho_\sigma], \quad (6)$$

$$\left( \frac{\ddot{a}}{a} \right) = -\frac{\kappa^2}{3} \left[ \left( \frac{3\gamma}{2} - 1 \right) \rho_\gamma + 2p_\Phi + 2p_\sigma + V_\Phi(R_\Phi) + V_\sigma(R_\sigma) \right], \quad (7)$$

$$\ddot{R}_\Phi + 3H\dot{R}_\Phi - \frac{\Omega_\Phi^2}{a^6 R_\Phi^3} - \frac{\partial V_\Phi(R_\Phi)}{\partial R_\Phi} = 0, \quad (8)$$

$$\ddot{R}_\sigma + 3H\dot{R}_\sigma - \frac{\Omega_\sigma^2}{a^6 R_\sigma^3} - \frac{\partial V_\sigma(R_\sigma)}{\partial R_\sigma} = 0. \quad (9)$$

In equations (6),(7), the energy density and pressure of the canonical and the phantom fields, are given by:

$$\begin{aligned} \rho_\Phi &= \frac{1}{2}(\dot{R}_\Phi^2 + \frac{\Omega_\Phi^2}{a^6 R_\Phi^2}) + V_\Phi(R_\Phi) \\ p_\Phi &= \frac{1}{2}(\dot{R}_\Phi^2 + \frac{\Omega_\Phi^2}{a^6 R_\Phi^2}) - V_\Phi(R_\Phi) \end{aligned} \quad (10)$$

and

$$\begin{aligned} \rho_\sigma &= -\frac{1}{2}(\dot{R}_\sigma^2 + \frac{\Omega_\sigma^2}{a^6 R_\sigma^2}) + V_\sigma(R_\sigma) \\ p_\sigma &= -\frac{1}{2}(\dot{R}_\sigma^2 + \frac{\Omega_\sigma^2}{a^6 R_\sigma^2}) - V_\sigma(R_\sigma). \end{aligned} \quad (11)$$

In addition, the effect of the “angular component” of the system is embedded in the radial equations (8),(9) as an effective term containing the constants  $\Omega_\Phi$  and  $\Omega_\sigma$ , which are determined by the values of the first integrals of motion [18]. Finally,  $H$  is Hubble parameter.

The equation of state for the  $O(N)$  quintom is:

$$w = \frac{p_\Phi + p_\sigma}{\rho_\Phi + \rho_\sigma} = \frac{\dot{R}_\Phi^2 - \dot{R}_\sigma^2 + \frac{1}{a^6} \left( \frac{\Omega_\Phi^2}{R_\Phi^2} - \frac{\Omega_\sigma^2}{R_\sigma^2} \right) - 2[V_\Phi(R_\Phi) + V_\sigma(R_\sigma)]}{\dot{R}_\Phi^2 - \dot{R}_\sigma^2 + \frac{1}{a^6} \left( \frac{\Omega_\Phi^2}{R_\Phi^2} - \frac{\Omega_\sigma^2}{R_\sigma^2} \right) + 2[V_\Phi(R_\Phi) + V_\sigma(R_\sigma)]}. \quad (12)$$

Thus, the constructed quintom model could produce a value  $w < -1$  if

$$\left| \dot{R}_\Phi^2 - \dot{R}_\sigma^2 + \frac{1}{a^6} \left( \frac{\Omega_\Phi^2}{R_\Phi^2} - \frac{\Omega_\sigma^2}{R_\sigma^2} \right) \right| < 2 |V_\Phi(R_\Phi) + V_\sigma(R_\sigma)|. \quad (13)$$

### 3 Stability analysis of the $O(N)$ quintom

We are interested in investigating the attractor properties of the  $O(N)$  quintom model, imposing exponential potentials, since they are known to be relevant in various cosmological models [20]. In particular we consider:

$$\begin{aligned} V_\Phi(R_\Phi) &= V_{\Phi_0} \exp(-\lambda\kappa R_\Phi) \\ V_\sigma(R_\sigma) &= V_{\sigma_0} \exp(-\lambda\kappa R_\sigma). \end{aligned} \quad (14)$$

Using this specific potential ansatz, the radial equations of motion (8),(9) become:

$$\ddot{R}_\Phi + 3H\dot{R}_\Phi - \frac{\Omega_\Phi^2}{a^6 R_\Phi^3} - \lambda\kappa V_\Phi(R_\Phi) = 0, \quad (15)$$

$$\ddot{R}_\sigma + 3H\dot{R}_\sigma - \frac{\Omega_\sigma^2}{a^6 R_\sigma^3} - \lambda\kappa V_\sigma(R_\sigma) = 0. \quad (16)$$

Furthermore, using the definitions for the energy densities and pressures (10),(11), the Friedmann equations (6),(7) can be re-written as:

$$\dot{H} = -\frac{\kappa^2}{2} \left( \rho_\gamma + p_\gamma + \dot{R}_\Phi^2 + \frac{\Omega_\Phi^2}{a^6 R_\Phi^2} - \dot{R}_\sigma^2 - \frac{\Omega_\sigma^2}{a^6 R_\sigma^2} \right) \quad (17)$$

$$H^2 = \frac{\kappa^2}{3} \left[ \rho_\gamma + \frac{1}{2} \left( \dot{R}_\Phi^2 + \frac{\Omega_\Phi^2}{a^6 R_\Phi^2} \right) + V_\Phi(R_\Phi) - \frac{1}{2} \left( \dot{R}_\sigma^2 + \frac{\Omega_\sigma^2}{a^6 R_\sigma^2} \right) + V_\sigma(R_\sigma) \right]. \quad (18)$$

Finally, the equations close by considering the evolution of the barotropic (matter) density:

$$\dot{\rho}_\gamma = -3H(\rho_\gamma + p_\gamma). \quad (19)$$

In order to perform the stability analysis of the  $O(N)$  quintom model, we have to transform the dynamical system (15)-(18) into an autonomous form [21]. This will be achieved by introducing the auxiliary variables:

$$\begin{aligned} x_\Phi &= \frac{\kappa}{\sqrt{6}H} \dot{R}_\Phi, & x_\sigma &= \frac{\kappa}{\sqrt{6}H} \dot{R}_\sigma \\ y_\Phi &= \frac{\kappa\sqrt{V_\Phi(R_\Phi)}}{\sqrt{3}H}, & y_\sigma &= \frac{\kappa\sqrt{V_\sigma(R_\sigma)}}{\sqrt{3}H} \\ z_\Phi &= \frac{\kappa}{\sqrt{6}H} \frac{\Omega_\Phi}{a^3 R_\Phi}, & z_\sigma &= \frac{\kappa}{\sqrt{6}H} \frac{\Omega_\sigma}{a^3 R_\sigma} \\ \xi_\Phi &= \frac{1}{\kappa R_\Phi}, & \xi_\sigma &= \frac{1}{\kappa R_\sigma}, \end{aligned} \quad (20)$$

together with  $M = \log a$ .

Using these variables, we result in the following autonomous system:

$$\begin{aligned}
\frac{dx_\Phi}{dM} &= \frac{3}{2}x_\Phi T_1 - 3x_\Phi + \sqrt{6} z_\Phi^2 \xi_\Phi + \sqrt{\frac{3}{2}} \lambda y_\Phi^2 \\
\frac{dx_\sigma}{dM} &= \frac{3}{2}x_\sigma T_1 - 3x_\sigma + \sqrt{6} z_\sigma^2 \xi_\sigma - \sqrt{\frac{3}{2}} \lambda y_\sigma^2 \\
\frac{dy_\Phi}{dM} &= \frac{3}{2}y_\Phi T_1 - \sqrt{\frac{3}{2}} \lambda x_\Phi y_\Phi \\
\frac{dy_\sigma}{dM} &= \frac{3}{2}y_\sigma T_1 - \sqrt{\frac{3}{2}} \lambda x_\sigma y_\sigma \\
\frac{dz_\Phi}{dM} &= \frac{3}{2}z_\Phi T_1 - 3z_\Phi - \sqrt{6} x_\Phi z_\Phi \xi_\Phi \\
\frac{dz_\sigma}{dM} &= \frac{3}{2}z_\sigma T_1 - 3z_\sigma - \sqrt{6} x_\sigma z_\sigma \xi_\sigma \\
\frac{d\xi_\Phi}{dM} &= -\sqrt{6} \xi_\Phi^2 x_\Phi \\
\frac{d\xi_\sigma}{dM} &= -\sqrt{6} \xi_\sigma^2 x_\sigma,
\end{aligned} \tag{21}$$

where  $T_1 = \gamma(1 - x_\Phi^2 - y_\Phi^2 - z_\Phi^2 + x_\sigma^2 - y_\sigma^2 + z_\sigma^2) + 2(x_\Phi^2 + z_\Phi^2 - x_\sigma^2 - z_\sigma^2)$ . Note also that the Friedmann equation (18) leads to the constraint equation:

$$x_\Phi^2 - x_\sigma^2 + y_\Phi^2 + y_\sigma^2 + z_\Phi^2 - z_\sigma^2 + \frac{\kappa^2 \rho_\gamma}{3H^2} = 1. \tag{22}$$

Finally, in terms of the auxiliary variables, the equation of state for the quintom (12) becomes:

$$w = \frac{x_\Phi^2 - x_\sigma^2 - y_\Phi^2 - y_\sigma^2 + z_\Phi^2 - z_\sigma^2}{x_\Phi^2 - x_\sigma^2 + y_\Phi^2 + y_\sigma^2 + z_\Phi^2 - z_\sigma^2}. \tag{23}$$

The critical points  $(x_{\Phi c}, x_{\sigma c}, y_{\Phi c}, y_{\sigma c}, z_{\Phi c}, z_{\sigma c}, \xi_{\Phi c}, \xi_{\sigma c})$  of the autonomous system (21) are obtained by setting the left hand sides of the equations to zero. The real and physically meaningful of them are presented in table 1.

In order to determine the stability properties of these critical points, we proceed as follows: We expand the auxiliary variables around the critical points as

$$\begin{aligned}
x_\Phi &= x_{\Phi c} + u_\Phi \\
x_\sigma &= x_{\sigma c} + u_\sigma \\
y_\Phi &= y_{\Phi c} + v_\Phi \\
y_\sigma &= y_{\sigma c} + v_\sigma \\
z_\Phi &= z_{\Phi c} + w_\Phi \\
z_\sigma &= z_{\sigma c} + w_\sigma \\
\xi_\Phi &= \xi_{\Phi c} + \chi_\Phi \\
\xi_\sigma &= \xi_{\sigma c} + \chi_\sigma.
\end{aligned} \tag{24}$$

Cr. Point	$x_{\Phi_c}$	$x_{\sigma_c}$	$y_{\Phi_c}$	$y_{\sigma_c}$	$z_{\Phi_c}$	$z_{\sigma_c}$	$\xi_{\Phi_c}$	$\xi_{\sigma_c}$
A	$+\sqrt{1+x_\sigma^2}$	$x_\sigma$	0	0	0	0	0	0
B	$-\sqrt{1+x_\sigma^2}$	$x_\sigma$	0	0	0	0	0	0
C	$\frac{\sqrt{6}}{\lambda}$	$+\frac{\sqrt{6-\lambda^2}}{\lambda}$	0	0	0	0	0	0
D	$\frac{\sqrt{6}}{\lambda}$	$-\frac{\sqrt{6-\lambda^2}}{\lambda}$	0	0	0	0	0	0
E	$\frac{\lambda}{\sqrt{6}}$	0	$+\sqrt{1-\frac{\lambda^2}{6}}$	0	0	0	0	0
F	$\frac{\lambda}{\sqrt{6}}$	0	$-\sqrt{1-\frac{\lambda^2}{6}}$	0	0	0	0	0
G	$\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}$	0	$+\sqrt{\frac{3}{2}}\frac{\sqrt{\gamma(2-\gamma)}}{\lambda}$	0	0	0	0	0
H	$\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}$	0	$-\sqrt{\frac{3}{2}}\frac{\sqrt{\gamma(2-\gamma)}}{\lambda}$	0	0	0	0	0
I	$+\frac{\sqrt{\lambda^2+6}}{\lambda}$	$\frac{\sqrt{6}}{\lambda}$	0	0	0	0	0	0
J	$-\frac{\sqrt{\lambda^2+6}}{\lambda}$	$\frac{\sqrt{6}}{\lambda}$	0	0	0	0	0	0
K	0	$-\frac{\lambda}{\sqrt{6}}$	0	$+\frac{\sqrt{\lambda^2+6}}{\sqrt{6}}$	0	0	0	0
L	0	$-\frac{\lambda}{\sqrt{6}}$	0	$-\frac{\sqrt{\lambda^2+6}}{\sqrt{6}}$	0	0	0	0

Table 1: The real and physically meaningful critical points of the autonomous system (21).

In expressions (24),  $u_\Phi, u_\sigma, v_\Phi, v_\sigma, w_\Phi, w_\sigma, \chi_\Phi, \chi_\sigma$  are just the perturbations of the variables near the critical points and we consider them forming a column vector denoted as  $\mathbf{U}$ . Inserting these expansions into the autonomous system (21), we can obtain the equations for the perturbations up to first order as:

$$\mathbf{U}' = \mathbf{Q} \cdot \mathbf{U}, \quad (25)$$

where the prime denotes differentiation with respect to  $M$ .

For the critical points  $(x_{\Phi_c}, x_{\sigma_c}, y_{\Phi_c}, y_{\sigma_c}, z_{\Phi_c}, z_{\sigma_c}, \xi_{\Phi_c}, \xi_{\sigma_c})$ , the coefficients of the perturbation equations form a  $8 \times 8$  matrix  $\mathbf{Q}$ . Thus, for each critical point of table 1, the eigenvalues of  $\mathbf{Q}$  determine the type and stability of this specific critical point. The explicit form of the matrix  $\mathbf{Q}$  and its eigenvalues for the critical points of table 1, i.e for the autonomous system (21), are given in the Appendix. In table 2 we present the results of the stability analysis. In addition, for each critical point we calculate the values of  $w$  (given by relation (23)), and of  $\frac{\kappa^2 \rho_\gamma}{3H^2}$  (given by the constraint (22)). Thus,  $\frac{\kappa^2 \rho_\gamma}{3H^2} = 0$  means that the universe is dominated completely by the quintom fields. In the next section we discuss the cosmological implications of the obtained results.

## 4 Cosmological implications

The critical points K and L are stable nodes for every  $\gamma$  and every  $\lambda$  and thus they constitute late-time attractors. Since  $\frac{\kappa^2 \rho_\gamma}{3H^2} = 0$  they correspond to quintom-dominated universes, with the quintom equation-of-state parameter being  $w = -1 - \frac{\lambda^2}{3}$ . The fact that  $x_{\Phi_c} = y_{\Phi_c} = 0$

Cr. Point	Existence	Stability	$w$	$\frac{\kappa^2 \rho_\gamma}{3H^2}$
A	$\forall \gamma$ and $\forall \lambda$	Stable node for $\lambda x_\sigma > \sqrt{6}$ Unstable node for $-\sqrt{6-\lambda^2} < \lambda x_\sigma < \sqrt{6-\lambda^2}$ Saddle point for $\lambda x_\sigma < -\sqrt{6-\lambda^2}$ or $\sqrt{6-\lambda^2} < \lambda x_\sigma < \sqrt{6}$	1	0
B	$\forall \gamma$ and $\forall \lambda$	Unstable node	1	0
C	$0 < \lambda^2 < 6$	Unstable node	1	0
D	$0 < \lambda^2 < 6$	Unstable node	1	0
E	$\lambda^2 < 6$	Saddle point	$-1 + \lambda^2/3$	0
F	$\lambda^2 < 6$	Saddle point	$-1 + \lambda^2/3$	0
G	$\forall \gamma$ and $\lambda \neq 0$	Saddle point	$\gamma - 1$	$1 - 3\gamma/\lambda^2$
H	$\forall \gamma$ and $\lambda \neq 0$	Saddle point	$\gamma - 1$	$1 - 3\gamma/\lambda^2$
I	$\forall \gamma$ and $\lambda \neq 0$	Saddle point	1	0
J	$\forall \gamma$ and $\lambda \neq 0$	Unstable node	1	0
K	$\forall \gamma$ and $\forall \lambda$	Stable node	$-1 - \lambda^2/3$	0
L	$\forall \gamma$ and $\forall \lambda$	Stable node	$-1 - \lambda^2/3$	0

Table 2: The properties of the critical points of the autonomous system (21).

implies that the phantom component of the quintom plays the main role in this solution. Thus, this solution corresponds to the phantom solution of the literature, with the expected  $w < -1$ . However, the more complex dynamics of the quintom model makes the system qualitative different from the simple phantom one, even in this particular solution where the canonical field seems to have a trivial contribution. Indeed, in the simple phantom case [19] a stable node exists only when  $\lambda^2 < 6$ , which in turn leads to a lower bound for  $w$ . As we see this is not true in the  $O(N)$  quintom paradigm, where the stable node is independent of the parameter values. Thus, although in the previous studies an accelerating universe was not always a late-time attractor, in the present analysis we show that such a cosmological era is always a stable equilibrium solution. In addition, the possible values of  $w$  are not bounded.

The critical points A and B are a novel property of the present quintom model. First of all, since for every  $x_\sigma$  there is always a  $x_\Phi$  which lead to zero left hand sides of the autonomous system (21) (given the nullification of the other variables), A and B corresponds to loci of critical points. They correspond to quintom-dominated solutions, with  $w = 1$ . Note that in this case both the canonical and the phantom field are relevant for the cosmological evolution. The critical point B is unstable. However, in the case  $\lambda x_\sigma > \sqrt{6}$ , the critical point A is a stable node. This behavior is different from both the simple canonical [20] and the simple phantom [19] fields.

The critical points C, D and I correspond to quintom-dominated solutions, and especially to solutions where the constraint (22) is dominated only by the kinetic energy of the canonical and the phantom fields. As expected, these “kinetic”-dominated solutions are unstable and are only expected to be relevant at early times. Furthermore, the critical point J corresponds to a quintom-dominated unstable solution.

The critical points G and H correspond to saddle points of the dynamical system at hand,



for every  $\gamma$  and every  $\lambda \neq 0$ . Note that although for a specific combination of these parameters ( $\gamma > \frac{2}{9}$  and  $\lambda^2 > \frac{24\gamma^2}{9\gamma-2}$ ) two of the eigenvalues form a complex conjugate pair with negative real part, the corresponding critical points are not stable spirals but saddle points, due to the presence of a positive eigenvalue. This is a crucial difference comparing to the simple canonical field [20] and reveals that the more complex quintom dynamics changes qualitatively the behavior of the system, even if the phantom field does not seem to play an important role in this case ( $x_{\sigma c} = y_{\sigma c} = 0$ ). The value of  $w$  for the critical points G and H is  $\gamma - 1$ , thus for the allowed  $\gamma$  values it corresponds to  $-1 < w < 1$ . In other words, this solution corresponds to the quintessence universe and the fact that is not a stable point, i.e. a late-time attractor, means that in the model at hand the universe cannot result in a quintessence-type solution with  $w > -1$ , even in the cases where the phantom field seems to be irrelevant. Finally, we mention that for these points  $\frac{\kappa^2 \rho_\gamma}{3H^2} = 1 - \frac{3\gamma}{\lambda^2}$ . Thus, although G and H exist for every  $\gamma$  and every  $\lambda \neq 0$ , the corresponding solution has a physical meaning only when  $\lambda^2 > 3\gamma$ . This fact was also mentioned in [20] for the simple canonical model.

The critical points E and F, are saddle points. They correspond to quintom-dominated universes, with  $w = -1 + \frac{\lambda^2}{3}$ . The fact that  $x_{\sigma c} = y_{\sigma c} = 0$  means that the phantom field is not relevant for this solution, and the cosmological evolution is driven by the canonical field. Thus, this solution also corresponds to the quintessence universe. However, while this solution is a stable node, i.e. a late-time attractor, for the simple quintessence model in the case  $\lambda^2 < 3\gamma$  [20], in the present quintom model it is always a saddle point. Therefore, the quintessence-type solution cannot be an equilibrium point for the cosmological evolution of the universe. Once again we see that the more complex  $O(N)$  quintom dynamics implies qualitatively new behavior for the cosmological evolution comparing to the previous simple models.

From the aforementioned investigation we conclude that the  $O(N)$  quintom has a variety of critical points. It is interesting to see that for not very steep potentials and/or a small  $x_\sigma$ , the attractor A disappears. Thus, the attractors K and L are the only ones for a very large area of the two-dimensional parameter space  $(\gamma, \lambda)$ . And since the initial value of  $w$  can be easily above  $-1$  we conclude that the transition through the  $w = -1$  divide has a large probability to be realized, as the system is attracted by the attractors K and L.

## 5 Conclusions

In the present work we investigated the quintom paradigm of dark energy in the generalized case where the corresponding canonical and phantom fields possess  $O(N)$  symmetries. Our analysis reveals that this  $O(N)$  quintom model exhibits novel properties comparing to the simple canonical and phantom scenarios. In particular, we find that the universe cannot result in a quintessence-type solution with  $w > -1$ , even in the cases where the phantom field seems to be irrelevant. On the contrary, there are always late-time attracting equilibrium points which correspond to quintom-dominated, accelerating universes with  $w < -1$ , and for a very large area of the parameter space they are the only ones. This is in contrast with the previous simple-quintom results, where an accelerating universe is a possible late-time stable solution but it is not guaranteed. In addition, the more complex quintom dynamics, refutes the lower bound of  $w$  obtained previously in the simple phantom paradigm, making its possible values

unlimited. Finally, the transition from above to below of the phantom divide  $w = -1$ , has a large probability to be realized. These features make the present  $O(N)$  quintom model consistent with cosmological observations.

## Appendix: stability of the critical points

For the critical points  $(x_{\Phi_c}, x_{\sigma_c}, y_{\Phi_c}, y_{\sigma_c}, z_{\Phi_c}, z_{\sigma_c}, \xi_{\Phi_c}, \xi_{\sigma_c})$ , the coefficients of the perturbation equations form a  $8 \times 8$  matrix  $\mathbf{Q}$ , which for the autonomous system (21) reads:

$$\begin{aligned}
\mathbf{Q}_{11} &= 3 \left[ -1 + \frac{T_2}{2} + x_{\Phi_c}^2 (2 - \gamma) \right] \\
\mathbf{Q}_{12} &= 3 x_{\Phi_c} x_{\sigma_c} (\gamma - 2) \\
\mathbf{Q}_{13} &= y_{\Phi_c} (\sqrt{6} \lambda - 3 \gamma x_{\Phi_c}) \\
\mathbf{Q}_{14} &= -3 \gamma x_{\Phi_c} y_{\sigma_c} \\
\mathbf{Q}_{21} &= 3 x_{\Phi_c} x_{\sigma_c} (2 - \gamma) \\
\mathbf{Q}_{22} &= 3 \left[ -1 + \frac{T_2}{2} + x_{\sigma_c}^2 (\gamma - 2) \right] \\
\mathbf{Q}_{23} &= -3 \gamma x_{\sigma_c} y_{\Phi_c} \\
\mathbf{Q}_{24} &= y_{\sigma_c} (-\sqrt{6} \lambda - 3 \gamma x_{\sigma_c}) \\
\mathbf{Q}_{31} &= \left[ -\sqrt{\frac{3}{2}} \lambda + 3 x_{\Phi_c} (2 - \gamma) \right] y_{\Phi_c} \\
\mathbf{Q}_{32} &= 3 x_{\sigma_c} y_{\Phi_c} (\gamma - 2) \\
\mathbf{Q}_{33} &= -\sqrt{\frac{3}{2}} \lambda x_{\Phi_c} + \frac{3 T_2}{2} - 3 \gamma y_{\Phi_c}^2 \\
\mathbf{Q}_{34} &= -3 \gamma y_{\Phi_c} y_{\sigma_c} \\
\mathbf{Q}_{41} &= 3 x_{\Phi_c} y_{\sigma_c} (2 - \gamma) \\
\mathbf{Q}_{42} &= \left[ -\sqrt{\frac{3}{2}} \lambda + 3 x_{\sigma_c} (\gamma - 2) \right] y_{\sigma_c} \\
\mathbf{Q}_{43} &= -3 \gamma y_{\Phi_c} y_{\sigma_c} \\
\mathbf{Q}_{44} &= -\sqrt{\frac{3}{2}} \lambda x_{\sigma_c} + \frac{3 T_2}{2} - 3 \gamma y_{\sigma_c}^2 \\
\mathbf{Q}_{55} &= 3 \left( -1 + \frac{T_2}{2} \right) \\
\mathbf{Q}_{66} &= 3 \left( -1 + \frac{T_2}{2} \right),
\end{aligned}$$

where  $T_2 = \gamma(1 - x_{\Phi_c}^2 + x_{\sigma_c}^2 - y_{\Phi_c}^2 - y_{\sigma_c}^2) + 2(x_{\Phi_c}^2 - x_{\sigma_c}^2)$ . All the other components of  $\mathbf{Q}$  are zero for the physically interested critical points of table 1.

The eigenvalues  $\nu_i$  ( $i = 1, \dots, 8$ ) of  $\mathbf{Q}$  for the specific critical points of table 1, i.e for the autonomous system (21), are presented in table 3, where  $T_3 = \frac{1}{\lambda^2} \sqrt{\lambda^2(2 - \gamma)(24\gamma^2 + 2\lambda^2 - 9\gamma\lambda^2)}$ .

Thus, by determining the sign of the real parts of these eigenvalues, we can classify the corresponding critical point [21].

Cr. Point	$\nu_1$	$\nu_2$	$\nu_3$	$\nu_4$	$\nu_5$	$\nu_6$	$\nu_7$	$\nu_8$
A	$3(2 - \gamma)$	$3 - \sqrt{\frac{3}{2}}\lambda x_\sigma$	$3 - \sqrt{\frac{3}{2}}\lambda\sqrt{1 + x_\sigma^2}$	0	0	0	0	0
B	$3(2 - \gamma)$	$3 - \sqrt{\frac{3}{2}}\lambda x_\sigma$	$3 + \sqrt{\frac{3}{2}}\lambda\sqrt{1 + x_\sigma^2}$	0	0	0	0	0
C	$6 - 3\gamma$	$3 - 3\sqrt{1 - \frac{\lambda^2}{6}}$	0	0	0	0	0	0
D	$6 - 3\gamma$	$3 + 3\sqrt{1 - \frac{\lambda^2}{6}}$	0	0	0	0	0	0
E	$\frac{\lambda^2}{2}$	$\frac{\lambda^2}{2} - 3$	$\frac{\lambda^2}{2} - 3$	$\frac{\lambda^2}{2} - 3$	$\frac{\lambda^2}{2} - 3$	$\frac{\lambda^2}{2} - 3\gamma$	0	0
F	$\frac{\lambda^2}{2}$	$\frac{\lambda^2}{2} - 3$	$\frac{\lambda^2}{2} - 3$	$\frac{\lambda^2}{2} - 3$	$\frac{\lambda^2}{2} - 3$	$\frac{\lambda^2}{2} - 3\gamma$	0	0
G	$\frac{3}{2}\gamma$	$\frac{3}{4}(\gamma - 2 - T_3)$	$\frac{3}{4}(\gamma - 2 + T_3)$	$\frac{3}{2}(\gamma - 2)$	$\frac{3}{2}(\gamma - 2)$	$\frac{3}{2}(\gamma - 2)$	0	0
H	$\frac{3}{2}\gamma$	$\frac{3}{4}(\gamma - 2 - T_3)$	$\frac{3}{4}(\gamma - 2 + T_3)$	$\frac{3}{2}(\gamma - 2)$	$\frac{3}{2}(\gamma - 2)$	$\frac{3}{2}(\gamma - 2)$	0	0
I	$6 - 3\gamma$	$3 - \sqrt{\frac{3}{2}}\sqrt{6 + \lambda^2}$	0	0	0	0	0	0
J	$6 - 3\gamma$	$3 + \sqrt{\frac{3}{2}}\sqrt{6 + \lambda^2}$	0	0	0	0	0	0
K	$-\frac{\lambda^2}{2}$	$-\frac{\lambda^2}{2} - 3$	$-\frac{\lambda^2}{2} - 3$	$-\frac{\lambda^2}{2} - 3$	$-\frac{\lambda^2}{2} - 3$	$-\lambda^2 - 3\gamma$	0	0
L	$-\frac{\lambda^2}{2}$	$-\frac{\lambda^2}{2} - 3$	$-\frac{\lambda^2}{2} - 3$	$-\frac{\lambda^2}{2} - 3$	$-\frac{\lambda^2}{2} - 3$	$-\lambda^2 - 3\gamma$	0	0

Table 3: The eigenvalues of the matrix  $\mathbf{Q}$  of the perturbation equations of the autonomous system (21).

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